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Random response of hysteretic single-degree-of-freedom systems subjected to non-white excitations

Y. Matsushima
University of Tsukuba, Sakura, Nihari, Ibaraki, Japan

ABSTRACT: The nonlinear random response of the single-degree-of freedom system with the slip-type hysteretic restoring force characteristic is dealt with. The non-white excitation which has the Markoffian spectrum is assumed as the model of earthquake-like disturbance. The attention is focussed on the time change of equivalent natural frequency and of the hysteretically dissipated energy. The approximate solutions for them are derived in closed forms on the basis of the theoretical investigation. They are compared with the digital estimates obtained from the Monte Carlo simulation. The agreements between the both are satisfactory over the wide ranges of related parameters.

1 INTRODUCTION

The seismic acceleration has, in general, the complicated waveform and, in many cases, is assumed as the random process. The white noise is often taken as the simplest idealization for it on the basis that the power spectral density of the seismic acceleration does not depend on the frequency in an approximate sense. This constant spectral density makes the mathematical treatement easy. There is a high possibility to get the analytical solution in a simple form. It is useful to know fundamental properties of nonlinear random response of structures subjected to the white excitation. Many studies having this objective have been already performed. The writer has also presented the paper with this purpose (Matsushima 1984).

In order to get the realistic response especially for structures with high frequencies, however, the spectrum which more fits in the property of the actual ground motion should be used. In this sense the Markoffian spectrum is adopted herein as the model of earthquake-like excitation. This is obtained as the output of the first-order linear filter due to the white input. The most unrealistic point of the white idealization lies in the fact that even the high frequency components have the constant spectral density. The filter to get the Markoffian spectrum is a low-pass one. The spectral density monotonously decreases as the frequency increases, which is more suitable for the model of seismic motion than the white spectrum. The fact that the Markoffian spectrum has no peak will make the estimation and the character of response simple. The above-mentioned advantages are reasons why the Markoffian spectrum is adopted herein.

The structure having the high frequency does not usually receive the great amount of energy from the seismic motion in the elastic stage. The reduction of equivalent frequency caused by the damage, however, makes the input energy greater, which results in the increase of damage and therefore in the further decrease of frequency in a progressive way. One of the purposes of this paper is to get fundamental information about such phenomenon.

The structure is idealized by the mass-spring system having single degree of freedom. The restoring force has the slip-type hysteretic characteristic. This hysteresis is chosen for analysis, because this is the representative of the brittle behavior of structures which is accompanied by the degrading stiffness.

The time changes of the equivalent natural frequency and the hysteretically dissipated energy are theoretically investigated with the proper assumptions. The solutions are expressed in the explicit forms. The validity of solutions are numerically verified by the Monte Carlo simulation.

2 MARKOFFIAN SPECTRUM

The spectral density function of the

Markoffian spectrum is expressed by

$$S(\omega) = \frac{S_0}{1+(\frac{\omega}{\omega_h})^2} , \qquad (1)$$

where $S(\omega)$ is referred to as the both-sided spectral density with the angular frequency ω ranging from $-\infty$ to ∞ as its dependent variable. $S(\omega)$ has two parameters S_0 and ω_h. S_0 is the spectral density where $\omega=0$. ω_h is the angular frequency, the spectral density at which becomes one half of S_0. ω_h is an index which prescribes the decreasing rate of density due to the increasing frequency. ω_h is also the half-power point of $S(\omega)$. This means the power included in the range of frequencies less than $|\omega_h|$ equals one half of the total power as indicated below.

$$\int_{-\omega_h}^{\omega_h} S(\omega)d\omega = \frac{1}{2}\int_{-\infty}^{\infty} S(\omega)d\omega . \qquad (2)$$

The total power ρ of the Markoffian spectrum is equal to

$$\rho = \int_{-\infty}^{\infty} S(\omega)d\omega = \pi\omega_h S_0 . \qquad (3)$$

This spectrum is depicted in Fig.1.

The random noise having the Markoffian spectrum is simply called "Markoffian noise" hereafter. The Markoffian noise approaches the white noise with the power density level S_0 under the condition $\omega_h \to \infty$.

The averaged velocity response spectrum for the undamped case $S_V(T,0)$ of the Markoffian noise becomes as a lower soild line drawn in Fig.2, where ω_h is taken as 8π (s^{-1}) as its typical value. This curve is obtained from the fact that $S_V(T,0)$ is approximately proportional to the square root of $S(\omega)$. The ordinate has the arbitrary scale. The Housner's averaged velocity response spectrum for the undamped case is also shown in the same figure as a dotted line (Housner 1959). They are much alike in appearance, although the latter has a gentle peak. The white spectrum shown as an upper dotted line is getting apart from the averaged character of seismic motion as the period becomes shorter. This trend has already pointed out in Section 1.

As is found from Eq.(1), $S(\omega)$ is nearly constant in relatively low frequencies, while proportional to ω^{-2} in high frequencies. This means $S_V(T,0)$ for the Markoffian noise is approximetely proportional to the period T in the range of short periods, while almost constant in long periods. In other words, the averaged acceleration

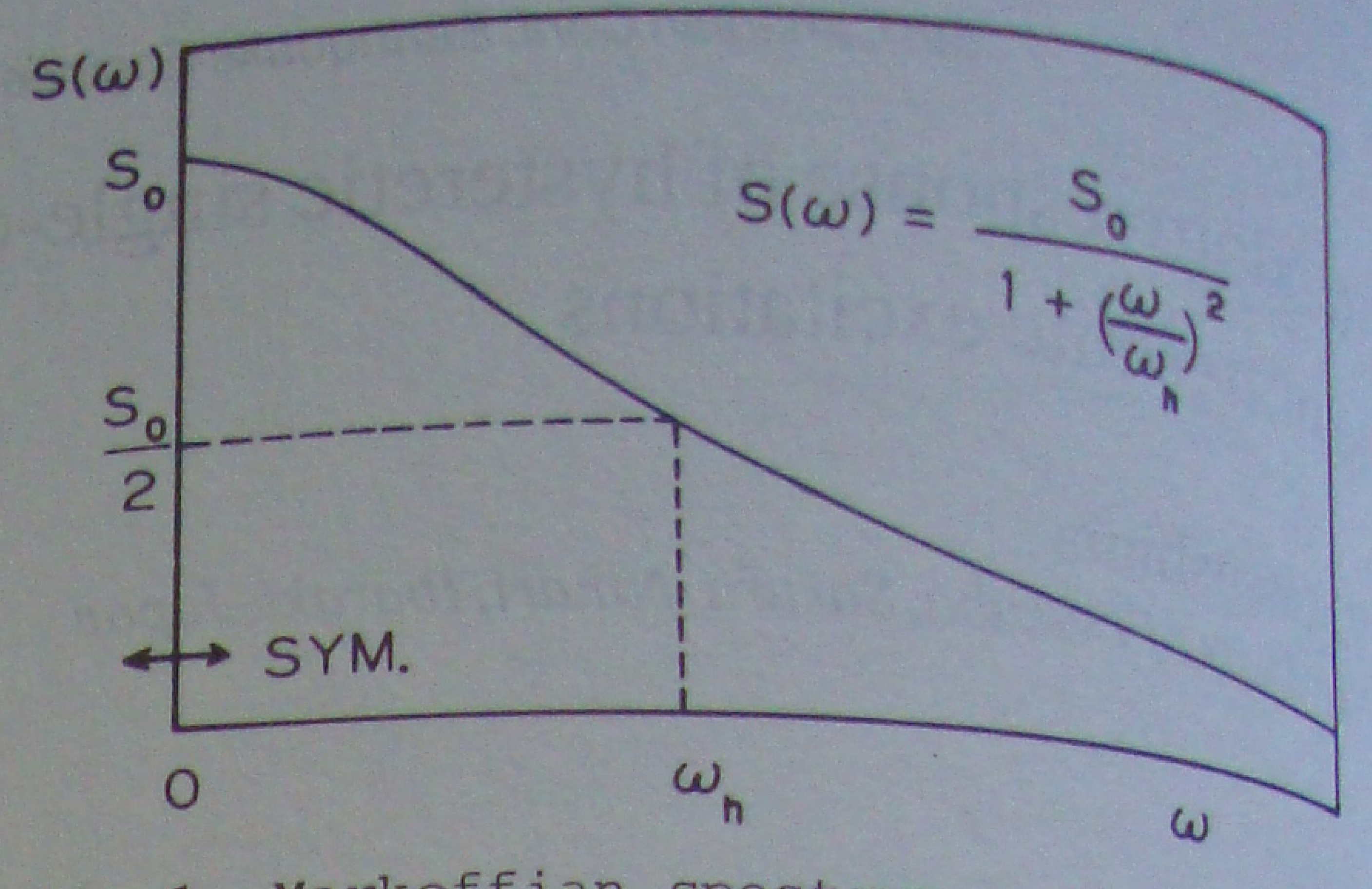


Fig.1 Markoffian spectrum

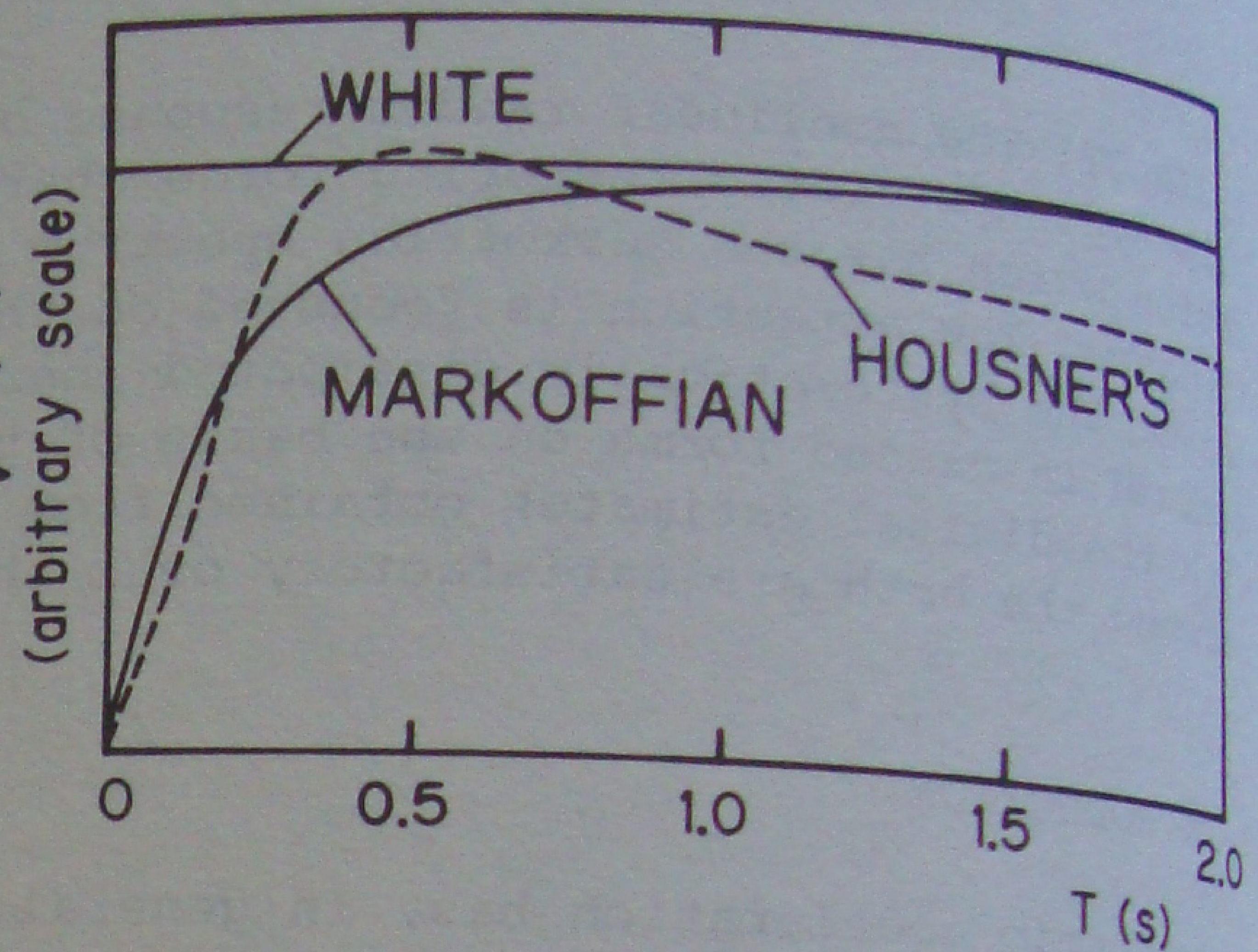


Fig.2 Velocity response spectra for undamped case

response spectrum for the undamped case $S_A(T,0)$ is almost constant in short periods and approximately proportional to T^{-1} in long periods. This is similar in appearance to "dynamic coefficient" R_t given in the Japanese Building Code. It indicates the Markoffian spectrum is appropriate as the model which characterizes the seismic motion as a whole.

3 INPUT-OUTPUT SYSTEM

It is assumed that the undamped single-degree-of-freedom system rested on the ground is suddenly subjected to the Markoffian noise which is taken as the ground acceleration. The equation of motion is given by

$$\ddot{x} + f(x,\dot{x}) = -U(t)N(t) , \qquad (4)$$

where x is the displacement of the system and $\cdot$ means the derivative with respect to time t. U(t) is the unit step function defined by

$$U(t) = \begin{cases} 0 & ; \ t < 0 \\ 1 & ; \ t \geqq 0 . \end{cases} \qquad (5)$$

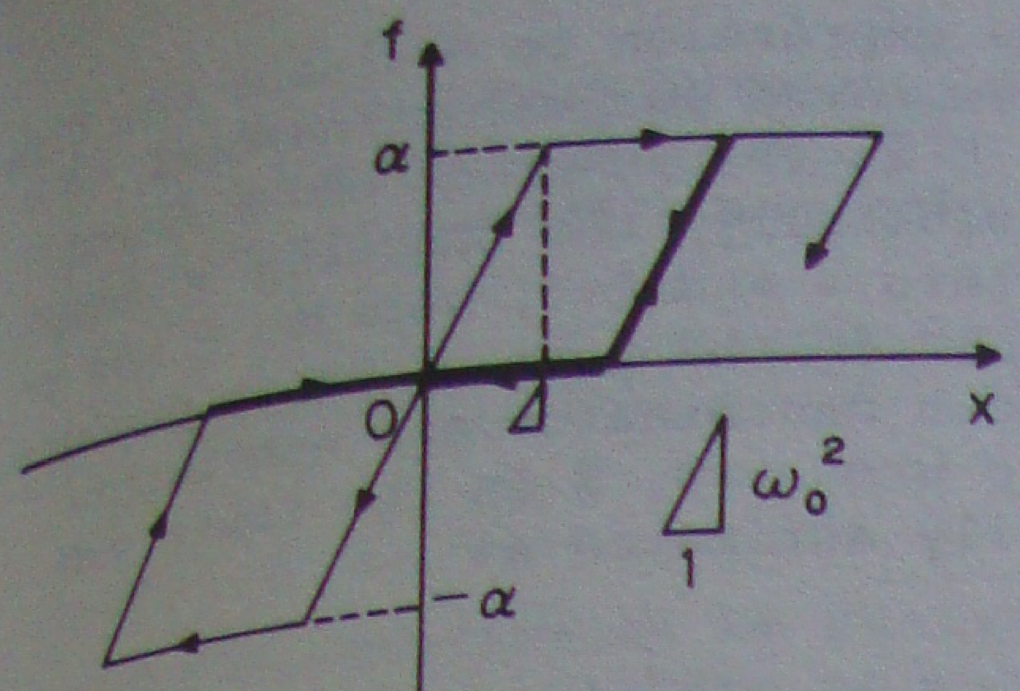


Fig.3 Slip-type hysteresis

N(t) denotes the stationary Markoffian noise.

$f(x,\dot{x})$ represents a restoring force function which has the slip-type hysteretic characteristic as illustrated in Fig.3. The plastic stiffness after yielding is taken as zero. The mass rested on the origin excurses on the lines with arrows indicated for example. The loops do not intersect mutually. when the restoring force becomes zero, the displacement progresses towards the last slip displacement without the change of restoring force. This is the typical pattern of the character of structures with X-type steel bracings. The timber and reinforced concrete shear walls also have the similar property where the slippage takes place more or less. α, Δ and ω_0 given in the figure are the yield acceleration, yield displacement and natural angular frequency in the elastic region, respectively. The following relation holds among them:

$$\alpha = \omega_0^2 \Delta \ . \qquad (6)$$

4 EXPECTED TIME OF ELASTIC LIMIT OF RESPONSE

The initial conditions of Eq.(4) are $x=\dot{x}=0$ when t=0. In the non-stationary response, the system behaves elastically in the early stage. The expected time the response arrives at the elastic limit, which is denoted by t_0, is approximately estimated as follows:

The Markoffian spectrum varies gently with frequency. The elastic response goes through the narrow-banded process with the expected natural angular frequency ω_0. Therefore it is almost equivalent to the response due to the white noise with the spectral density $S(\omega_0)$. The expected maximum absolute value of displacement response of the undamped linear single-degree-of-freedom system under such white noise can be written as

$$|\bar{x}|_{max} = a\sqrt{\pi S(\omega_0) t}/\omega_0 \ , \qquad (7)$$

where a is the coefficient by which the standard deviation is multiplied to get the maximum value. Rosenblueth and Bustamante (1962) have estimated the value of a as

$$a = 2.348$$

It is considered that the expected time when $|\bar{x}|_{max}$ reaches the yield displacement α/ω_0^2 is equal to t_0. Hence the following equation holds:

$$t_0 = \frac{\alpha^2}{a^2 \pi S(\omega_0)\omega_0^2} \ . \qquad (8)$$

Now it is convenient to define the following nondimensional quantities:

$$\tau_0 \equiv \frac{t_0}{T_0} = \frac{\omega_0 t_0}{2\pi} \ , \qquad (9)$$

$$\xi \equiv \frac{\omega_0 S_0}{\alpha^2} \qquad (10)$$

and

$$\nu \equiv \frac{\omega_0}{\omega_h} \ . \qquad (11)$$

τ_0 is t_0 normalized by the natural period T_0 in the elastic stage. ξ stands for the nondimensional input intensity. ν is the ratio of the natural angular frequency in the elastic stage to the half-power point of the Markoffian spectrum.

The substitution of Eq.(1) into the right side of Eq.(8) and the application of Eqs.(9), (10) and (11) to Eq.(8) gives the following nondimensional time of the elastic limit:

$$\tau_0 = \frac{1+\nu^2}{2a^2\pi^2\xi} \ . \qquad (12)$$

5 TIME CHANGE OF EXPECTATION OF EQUIVALENT NATURAL FREQUENCY

The system goes into the plastic region when t is greater than t_0, and the apparent natural frequency changes. The amount of this change of frequency will be estimated as follows:

The expected cumulative plastic deformation per unit time is approximately equal to $\pi S(\omega_e)/(2\alpha)$ in the case of slip-type hysteresis shown in Fig.3. Here ω_e represents the expected equivalent natural angular frequency, which is assumed to exist even in the nonlinear response. The process is not necessarily narrow-banded in the inelastic response. The approximation comes also from this fact. The plastic deformation is defined by the deformation which excurses on the lines $f=\pm\alpha$. The cumulative plastic deformation is the sum of the absobute value of the plastic deformation.

The absolute value of velocity, when slipping on the x-axis in a free vibration with initial conditions $f=\pm\alpha$ and $\dot{x}=0$, equals α/ω_0. Since the slip deformation is considered equal to the plastic deformation, the time derivative of the expected equivalent natural period T_e becomes

$$\frac{dT_e}{dt'} = 4 \cdot \frac{\pi S(\omega_e)}{2\alpha} \cdot \frac{\omega_0}{\alpha} \ , \qquad (13)$$

where $t'=t-t_0$, which is the time measured from when the system goes beyond the elastic limit.

Since $T_e=2\pi/\omega_e$, Eq.(13) can be written as

$$\frac{d\omega_e}{dt'} = -\frac{\omega_0 \omega_e{}^2 S(\omega_e)}{\alpha^2} \ . \qquad (14)$$

Now the following nondimensional quantities are newly defined:

$$\tau' \equiv \frac{t'}{T_0} \qquad (15)$$

and

$$\beta \equiv \frac{\omega_e}{\omega_0} \ . \qquad (16)$$

β is the ratio of the expected equivalent natural angular frequency to the elastic one.

The application of Eqs.(1),(10),(11),(15) and (16) to Eq.(14) gives the following nondimensional form:

$$\frac{d\beta}{dt'} = -\frac{2\pi\xi\beta^2}{1+\nu^2\beta^2} \ . \qquad (17)$$

The transformation of Eq.(17) to

$$d\tau' = -\frac{1+\nu^2\beta^2}{2\pi\xi\beta^2}\, d\beta$$

and integration of both sides of above equation gives:

$$\tau' = \frac{(1-\beta)(1+\nu^2\beta)}{2\pi\xi\beta} \ . \qquad (18)$$

Solving Eq.(18) for β, there is obtained

$$\beta = \frac{1}{2\nu^2}\left(\nu^2-2\pi\xi\tau'-1+\sqrt{(\nu^2-2\pi\xi\tau'-1)^2+4\nu^2}\right) \ . \qquad (19)$$

This is the expression to evaluate the time change of expected equivalent natural frequency. β is the function of ν, ξ and τ', but essensially, the function of only two parameters ν and $\xi\tau'$, because ξ and τ' always appear in the form of their product. Although both denominator and numerator become zero when $\nu=0$, β in the limit $\nu\to 0$ can be easily estimated as

$$\beta = \frac{1}{1+2\pi\xi\tau'} \ . \qquad (20)$$

This is the time change of expected equivalent natural frequency under white input. Equaction (20) is also obtained directly by assigning zero to ν in Eq.(18).

6 TIME CHANGE OF EXPECTATION OF HYSTERETICALLY DISSIPATED ENERGY

The expectation $\bar{E}$ of the hysteretically dissipated energy E is zero, when $t \leqq t_0$. Since it is possible to consider that the energy given to the system is consumed exclusively by the hysteresis when $t \geqq t_0$, the following equation will hold according to the same reason described in Section 5:

$$\frac{d\bar{E}}{dt'} = \pi S(\omega_e) \ . \qquad (21)$$

Here the following quantity is defined in order to make $\bar{E}$ be dimensionless:

$$\bar{\lambda} \equiv \frac{\bar{E}}{\alpha\Delta} = \frac{\omega_0{}^2\bar{E}}{\alpha^2} \ . \qquad (22)$$

$\bar{\lambda}$ stands for the expectation of cumulative ductility factor λ which is defined as the cumulative plastic deformation devided by the yield displacement.

The application of Eqs.(1),(10),(11),(15), (16) and (22) to Eq.(21) gives the following nondimensional expression when $S(\omega)$ is the Markoffian spectrum:

$$\frac{d\bar{\lambda}}{d\tau'} = \frac{2\pi^2\xi}{1+\nu^2\beta^2} \ . \qquad (23)$$

Dividing both sides of Eq.(23) by the corresponding sides of Eq.(17), one gets

$$\frac{d\bar{\lambda}}{d\beta} = -\frac{\pi}{\beta^2} \ . \qquad (24)$$

Multiplying both sides of this equation by $d\beta$ and integrating, there is obtained

$$\bar{\lambda} = \frac{\pi(1-\beta)}{\beta} \ . \qquad (25)$$

This is the relation between $\bar{\lambda}$ and β. Substituting β given by Eq.(19) into the right side of Eq.(25), one gets

$$\bar{\lambda} = \frac{\pi}{2}(-\nu^2+2\pi\xi\tau'-1+\sqrt{(\nu^2-2\pi\xi\tau'-1)^2+4\nu^2}) \ . \tag{26}$$

This is the expression to evaluate the time change of the expectation of cumulative ductility factor. $\bar{\lambda}$ is also the function of ν, ξ and τ', but essentially the function of only ν and ξτ' as in the case of β.
Assigning zero to ν in Eq.(26), one has

$$\bar{\lambda} = 2\pi^2\xi\tau' \ . \tag{27}$$

This is identical with the expression already obtained as the expectation of cumulative ductility factor due to the white noise (matsushima, 1980).
The inelastic behavior of the system subjected to the Markoffian noise is illustrated in Fig.4. If the system property is linear, the natural angular frequency ω_0 is kept constant. The response is almost equivalent to that due to the white noise having the spectral density $S(\omega_0)$. The lower dotted line in the figure represents this case. If the system has the nonlinear property, however, the apparent frequency is decreased in the inelastic stage. The energy driven to the system increases as indicated by an arrow. The response becomes equivalent to that due to the white noise with the level S_0, after time elapses to some extent. The upper dotted line represents this case. The position of ω_0 in the figure corresponds to the case ν=2, and therefore $S(\omega_0)$ is 1/5 of S_0. The difference between these two levels, however, rapidly becomes much less after the system goes into the inelastic region.
This situation is illustrated from the viewpoint of the time change of cumulative ductility factor in Fig.5. The figure is also drawn for the case ν=2. The abscissa is the product of τ and ξ. The slope of $\bar{\lambda}$ when $\tau=\tau_0$ is given by

$$\left|\frac{d\bar{\lambda}}{d\tau'}\right|_{\tau'=0} = \frac{2\pi^2\xi}{1+\nu^2} \ , \tag{28}$$

as is found form Eq.(23). This value devided by ξ is shown as the slope of the lower dotted line. The lower white noise indicated in Fig.4 corresponds to this dotted line. If the system property is linear, $\bar{\lambda}$ increases along this line. However, since the system has the nonlinear character, $\bar{\lambda}$ increases along the solid line with the slope approaching that of the upper dotted line. The latter is obtained by

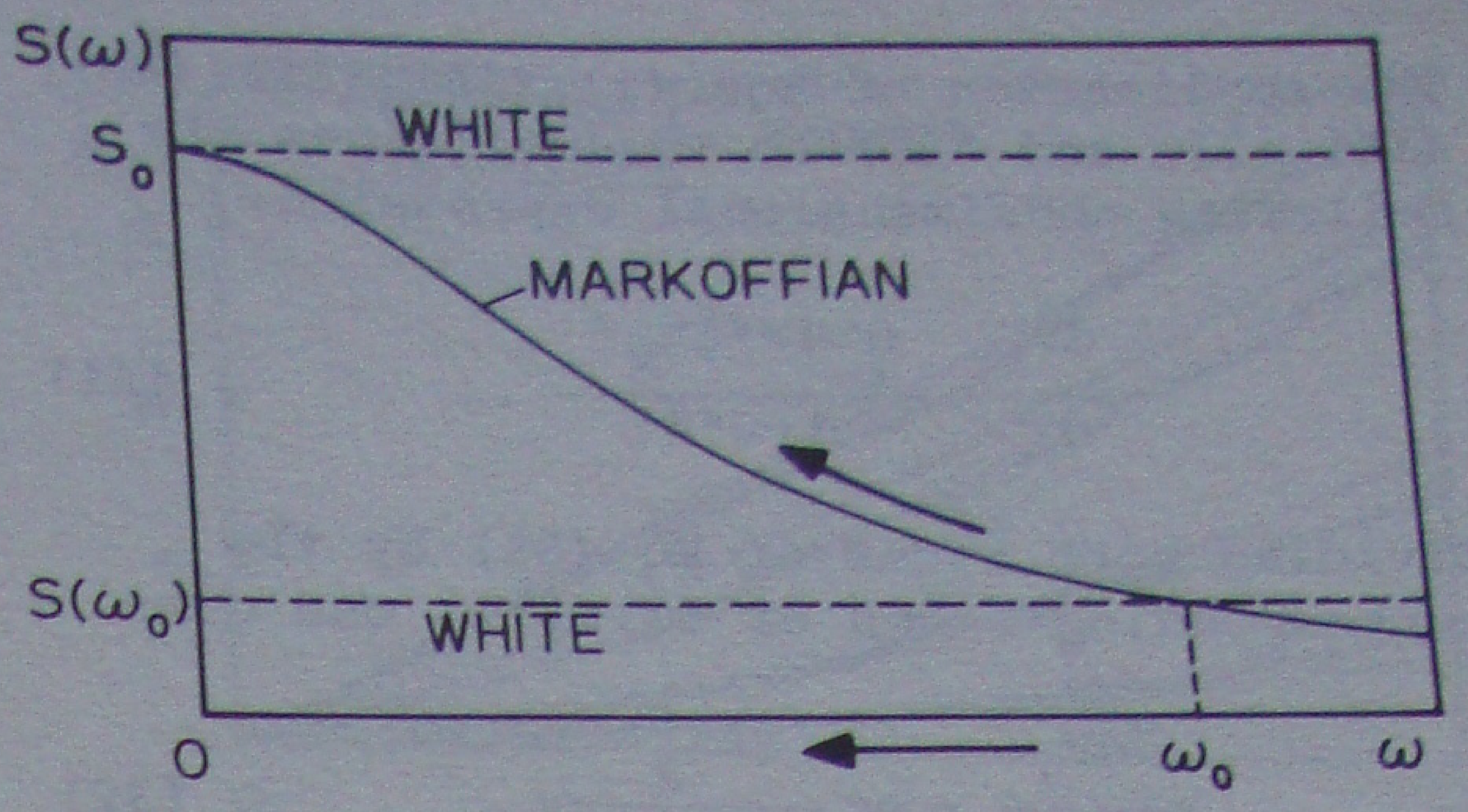


Fig.4 Nonlinear response due to white and Markoffian excitations

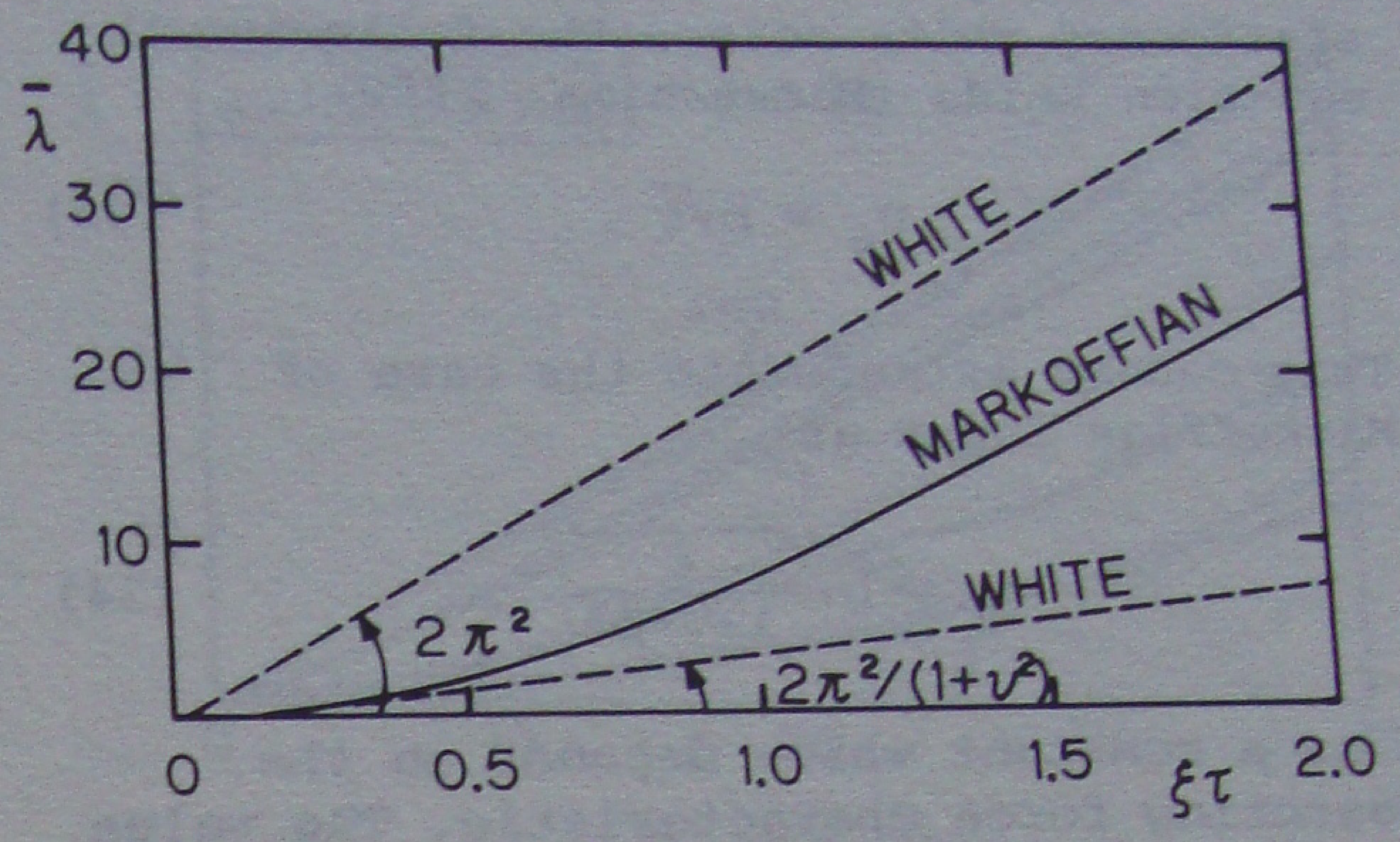


Fig.5 Time change of cumulative ductility factor

assinging zero to ν in Eq.(28). The upper white noise indicated in Fig.4 corresponds to this case. The solid line is obtained by assigning 2 to ν in Eq.(26). It is recognized that Eq.(26) reasonably expresses the nonlinearity of relevant hysteretic system and the character of Markoffian spectrum.

7 TIME CHANGE OF VARIANCE OF HYSTERETICALLY DISSIPATED ENERGY

The variance V_E of the hysteretically dissipated energy can be assumed as zero, when $t \leq t_0$. It is possible to write, like in the case of $\bar{E}$, as

$$\frac{dV_E}{dt'} = 2\pi S(\omega_e)\sigma_{\dot{x}}^2 \ , \tag{29}$$

when $t \geq t_0$. Here $\sigma_{\dot{x}}$ represents the standard deviation of the response velocity $\dot{x}$. The following quantity is defined to make $\sigma_{\dot{x}}$ be dimensionless:

$$\eta_{\dot{x}} = \frac{\sigma_{\dot{x}}}{\omega_0\Delta} \ . \tag{30}$$

The application of Eqs.(1),(10),(11),(15),(16),(22) and (30) to Eq.(29) gives the following nondimensional expression:

$$\frac{dV_\lambda}{d\tau'} = \frac{4\pi^2\xi\eta_{\dot{x}}^2}{1+\nu^2\beta^2} . \quad (31)$$

Dividing both sides of Eq.(31) by the corresponding sides of Eq.(17), one gets

$$\frac{dV_\lambda}{d\beta} = -\frac{2\pi\eta_{\dot{x}}^2}{\beta^2} . \quad (32)$$

When the system is subjected to the stationary white noise, the following equation holds (Matsushima, 1980):

$$\eta_{\dot{x}} = b\sqrt{\xi} . \quad (33)$$

This can be extended to the case of Markoffian input as

$$\eta_{\dot{x}} = b\sqrt{\frac{\xi}{1+\nu^2\beta^2}} . \quad (34)$$

b is a constant which depends on the restoring force characteristic. The value of b will be determined numerically, since it is quite difficult to obtain it in an analytical way.

Substitution of Eq.(34) into the right side of Eq.(32) gives

$$\frac{dV_\lambda}{d\beta} = -\frac{2\pi b^2\xi}{\beta^2(1+\nu^2\beta^2)} . \quad (35)$$

Multiplying both sides of Eq.(35) by $d\beta$ and integrating, one gets

$$V_\lambda = 2\pi b^2\xi\left[\frac{1-\beta}{\beta} - \nu(\tan^{-1}\nu - \tan^{-1}\nu\beta)\right] . \quad (36)$$

This is the relation between V_λ and β. The expression of V_λ against τ' is obtained by applying β given by Eq.(19) to the right side of Eq.(36). V_λ is the function of ν, ξ and τ'. In this case ξ appears individually in the expression of V_λ.

Assigning zero to ν in Eq.(36), one has

$$V_\lambda = 2\pi b^2\xi \cdot \frac{1-\beta}{\beta} . \quad (37)$$

Substitution of β given by Eq.(20) into the right side of this equation gives

$$V_\lambda = 4\pi^2 b^2\xi^2\tau' . \quad (38)$$

This is identical with the expression already obtained as the variance of cumulative ductility factor due to the white noise (Matsushima, 1980).

8 VERIFICATION BY MONTE CARLO SIMULATION

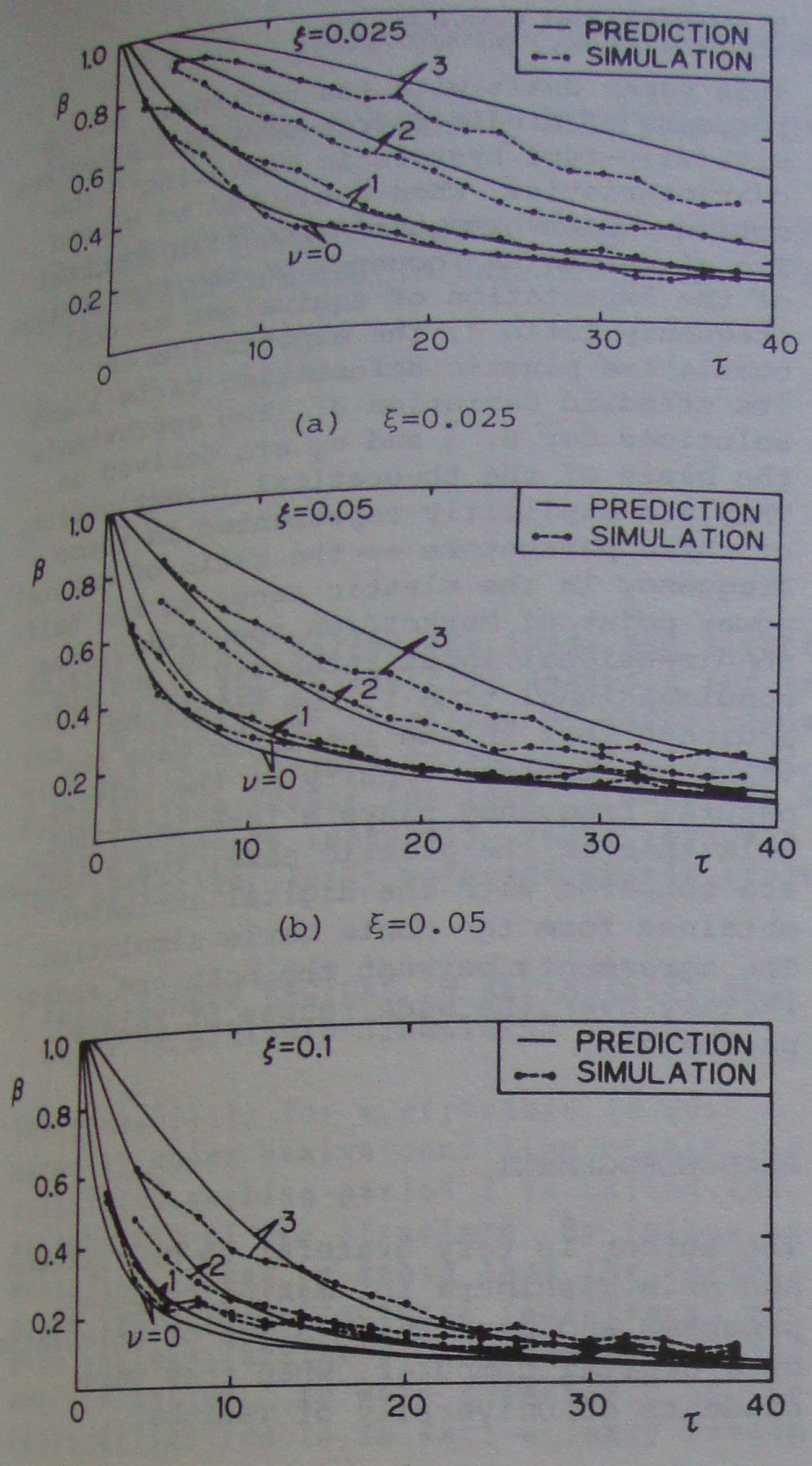
The digital simulation has been performed in order to verify analytical expressions obtained above. Fifty sample functions having the Markoffian spectrum have been generated on the basis of stochastic concept. The nonstationary nonlinear responses due to them have been numerically computed. The statistical treatment has been made on the results.

Since the expectation of equivalent natural frequency ratio β, the expectation of cumulative ductility factor $\bar{\lambda}$ and its variance V_λ are functions of ν, ξ and τ, β-τ, $\bar{\lambda}$-τ and V_λ-τ relations have been investigated with parameters ν and ξ. τ is the nondimensional time defined by t/T_0. The values of ν have been taken as 0,1,2 and 3. The case ν=0 corresponds to the white noise, and therefore the spectral density at the natural frequency is equal to S_0 from the outset. The initial values of spectral density for cases ν=1,2 and 3 are respectively 1/2, 1/5 and 1/10 of S_0. The values of ξ have been taken as 0.025, 0.05 and 0.1.

The β-τ relations with a parameter ν are displayed in Figs.6 (a)-(c) for three different ξ's. The soild lines represent analytical solutions given by Eq.(19). The dotted lines with points close to them stand for associated simulation estimates. The equivalent natural frequencies in the digital simulation have been evaluated by counting the number of times the displacement response crosses the time axis. The frequency corresponding to the number of crossing within the time interval of four units of nondimensional time τ have been assumed as the equivalent natural frequency at the central time of the relevant interval. The time intervals have been moved successively by two units of τ and associated numbers of crossing have been counted.

It is understood from these figures how β decreases as $\xi\tau$ increases. The dependence of β on ν is caused by the difference of associated initial values of spectral density. The dependance diminishes as time elapses because of the increase of spectral density. The degree of agreements between analytical solutions and digital estimates is generally acceptable, although not so excellent when ν is great.

Figures 7 (a)-(c) indicate $\bar{\lambda}$-τ relations in the same manner as in previous figures. The solid lines represent Eq.(26). The

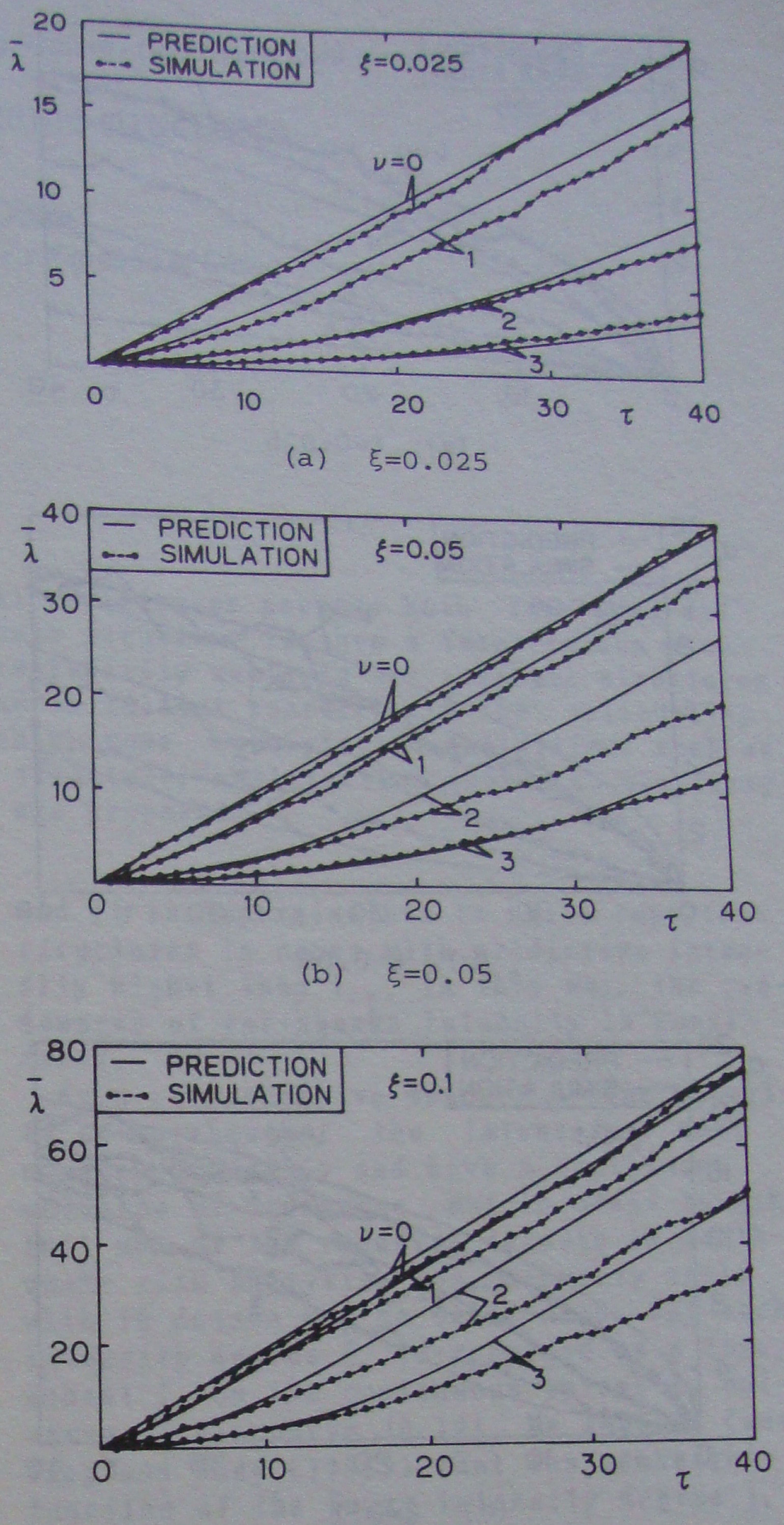


(a) ξ=0.025

(b) ξ=0.05

(c) ξ=0.1

Fig.6 Time change of expectation of equivalent natural frequency



(a) ξ=0.025

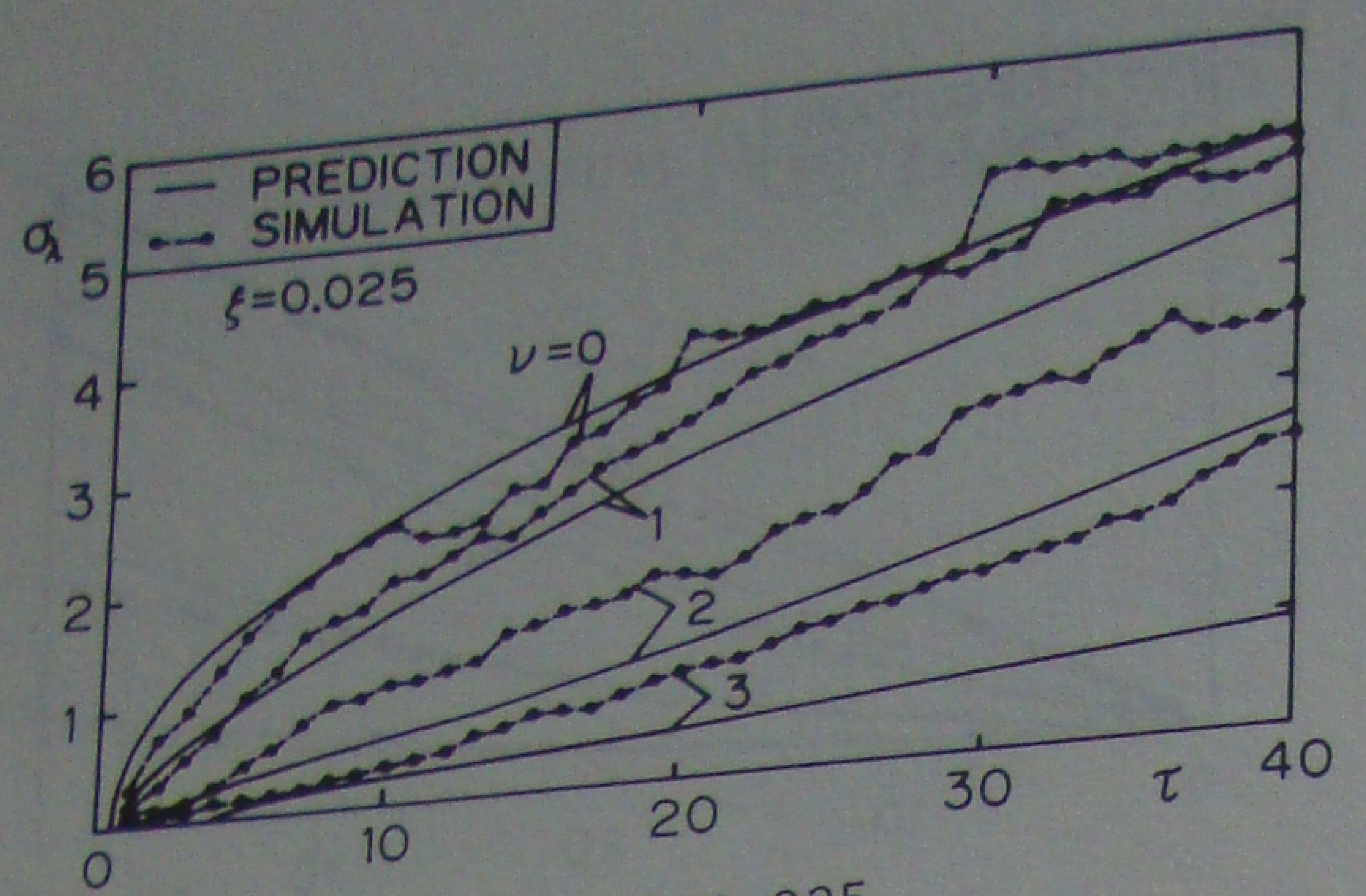
(b) ξ=0.05

(c) ξ=0.1

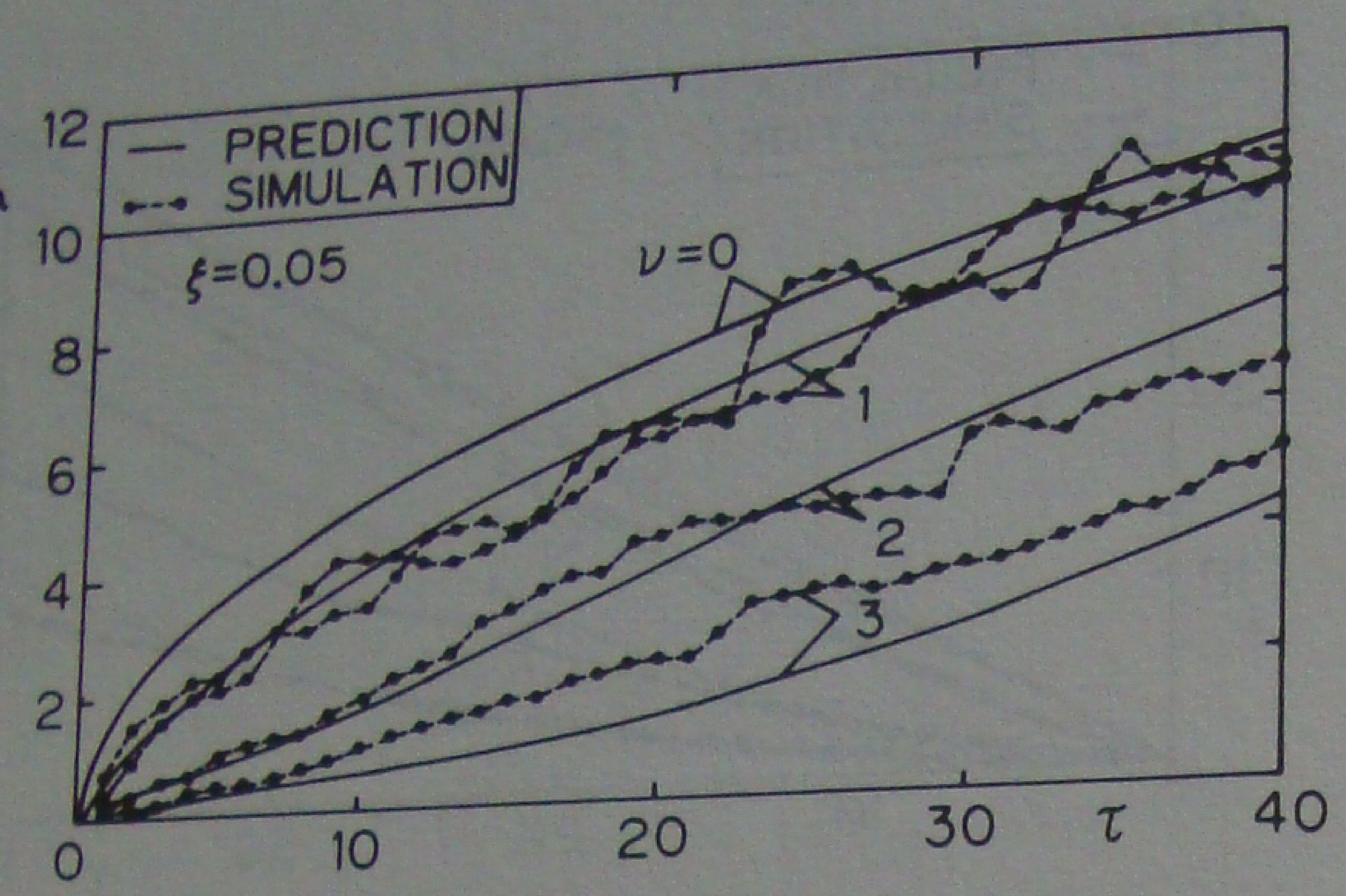
Fig.7 Time change of expectation of cumulative ductility factor

expected cumulative ductility factors in the digital simulation have been estimated from the ensemble average of sample cumulative ductility factors obtained by every one unit of nondimensional time τ. $\bar{\lambda}$ monotonously increases as $\xi\tau$ increases. $\bar{\lambda}$ is getting less as ν increases, but the difference due to ν diminishes with increasing time. This trend is similar to in the case of β, and has been already explained in Section 6. It is found that analytical solutions agree well with simulation estimates.

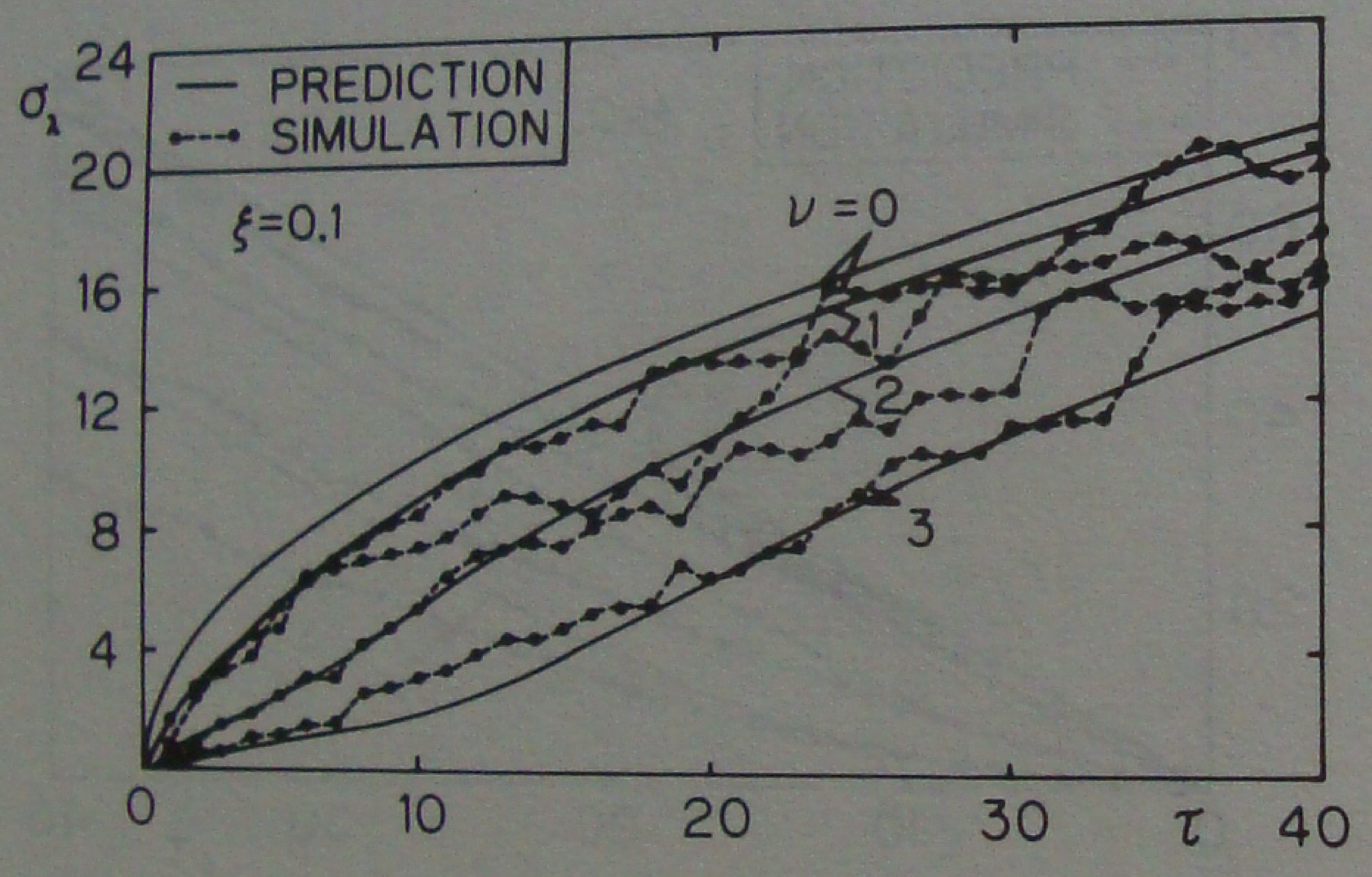
Figures 8 (a)-(c) indicate σ_λ-τ relations. σ_λ is the standard deviation of λ which is equal to $\sqrt{V_\lambda}$. σ_λ in the digital simulation has been calculated by the same way as in the case of $\bar{\lambda}$. The value of b appearing in the approximate solution has been estimated from the simulation results of the case of white input. The most probable value has been evaluated by applying tre least square method to the part of τ lying between 10 and 40 which is considered as stationary. As the results, 5.24, 5.64 and 4.72 have been obtained as the values of b for ξ of 0.025, 0.05 and 0.1, respectively. Since those values are relatively close to each other, the simple average has been taken as b as follows:



(a) ξ=0.025



(b) ξ=0.05



(c) ξ=0.1

Fig.8 Time change of standard deviation of cumulative ductility foctor

$$b = 5.20$$

σ_λ has been evaluated by Eq.(36) with this value to get the solid lines in these Figures. Since the number of samples is not sufficient enough to estimate the variance, there exists fluctuation in the simulation estimates. It is possible to say, however, that approximate solutions agree rather well with simulation estimates in these cases, too.

9 SUMMARY AND CONCLUSIONS

This paper deals with the nonlinear random response of single-degree-of-freedom systems with slip-type hysteretic restoring force characteristics, when subjected to ground motions which have the Markoffian spectra. The attention is focussed on the time change of the expectation of equivalent natural frequency ratio β, the expectation of cumulative plastic deformation ratio $\bar{\lambda}$ and its standard deviation σ_λ. The approximate solutions for β, $\bar{\lambda}$ and σ_λ are derived on the basis of the theoretical investigation. They are explicitly represented in terms of three parameters —— the ratio of natural frequency in the elastic range to the half-power point of Markoffian spectrum ν, the nondimensional input intensity ξ and the nondimensional time τ. The solutions indicate that in the inelastic case the value of spectral density at the initial natural frequency plays a less important role than in the elastic case. The solutions are compared with the digital estimates obtained form the Monte Carlo simulation. The agreements between the both are satisfactory over the wide ranges of related parameters.

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